Correction IE - Suite

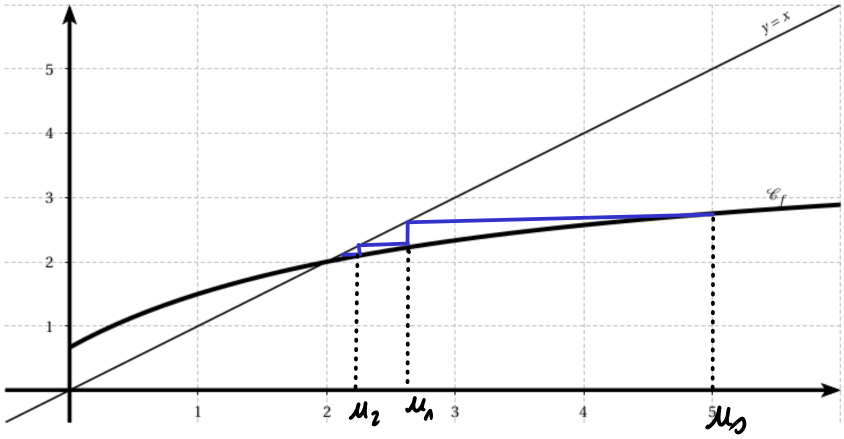
CPES - Géné

Exercice 1

$$\begin{cases} u_0 = 5 \\ \forall n \in \mathbb{N},\ u_{n+1} = \dfrac{4u_n + 2}{u_n + 3} \end{cases}$$

1°) $u_1 = \dfrac{22}{8} = \dfrac{11}{4}$

2°)



3°) $v_n = \dfrac{u_n - 2}{u_n + 1}$

Soit $n \in \mathbb{N}$

$$v_{n+1} = \frac{u_{n+1} - 2}{u_{n+1} + 1} = \frac{\dfrac{4u_n+2}{u_n+3} - \dfrac{2(u_n+3)}{u_n+3}}{\dfrac{4u_n+2}{u_n+3} + \dfrac{u_n+3}{u_n+3}} = \frac{2u_n - 4}{5u_n + 5}$$

donc $v_{n+1} = \dfrac{2(u_n-2)}{5(u_n+1)} = \dfrac{2}{5} \times \dfrac{u_n - 2}{u_n + 1} = \dfrac{2}{5} v_n$

Donc (v_n) est géométrique de raison $\dfrac{2}{5}$ avec $v_0 = \dfrac{5-2}{5+1} = \dfrac{3}{6} = \dfrac{1}{2}$

Ainsi $\forall n \in \mathbb{N}$, $v_n = \dfrac{1}{2} \times \left(\dfrac{2}{5}\right)^n$.

c) $v_n = \dfrac{u_n - 2}{u_n + 1} \iff v_n \times (u_n + 1) = u_n - 2$

$\iff u_n v_n - u_n = -2 - v_n$

$\iff u_n(v_n - 1) = -2 - v_n$

$\iff u_n = \dfrac{v_n + 2}{1 - v_n}$

d) $\forall n \in \mathbb{N}$, $u_n = \dfrac{\frac{1}{2} \times \left(\frac{2}{5}\right)^n + 2}{1 - \frac{1}{2} \times \left(\frac{2}{5}\right)^n} = \dfrac{\left(\frac{2}{5}\right)^n + 4}{2 - \left(\frac{2}{5}\right)^n} = \dfrac{2^n + 4 \times 5^n}{2 \times 5^n - 2^n}$

Exercice 2 $\quad u_0 = 0 \quad u_1 = 1 \quad \forall n \in \mathbb{N},\ u_{n+2} = 5u_{n+1} - 4u_n.$

1°) $u_2 = 5u_1 - 4u_0 = 5$

$u_3 = 5u_2 - 4u_1 = 5\times 5 - 4 = 21.$

2°) Première approche :

a) $v_n = u_{n+1} - u_n$

Soit $n \in \mathbb{N}$, $v_{n+1} = u_{n+2} - u_{n+1} = 5u_{n+1} - 4u_n - u_{n+1} = 4u_{n+1} - 4u_n$

$= 4(u_{n+1} - u_n)$

$v_{n+1} = 4v_n$

Donc (v_n) est géométrique de raison 4, de premier terme $v_0 = 1$

$\forall n \in \mathbb{N},\ v_n = 4^n$

b) $w_n = u_{n+1} - 4u_n$

Soit $n \in \mathbb{N}$, $w_{n+1} = u_{n+2} - 4u_{n+1} = 5u_{n+1} - 4u_n - 4u_{n+1}$

$= u_{n+1} - 4u_n$

$= w_n$

Donc (w_n) est stationnaire, $w_0 = 1$

Ainsi $\forall n \in \mathbb{N},\ w_n = 1$

c) $\left.\begin{array}{l} v_n = u_{n+1} - u_n \\ w_n = u_{n+1} - 4u_n \end{array}\right\}$ donc $v_n - w_n = 3u_n$

On a donc, $\forall n \in \mathbb{N}$, $u_n = \dfrac{v_n - w_n}{3} = \dfrac{4^n - 1}{3}$

2°) Seconde approche

a) $\forall n \in \mathbb{N},\ u_{n+2} = 5u_{n+1} - 4u_n$

Soit (t_n) défini par $t_0 = 0$ et $t_{n+1} = 4t_n + 1$, $t_1 = 1$

Alors $t_{n+2} = 4t_{n+1} + 1 = 4(4t_n + 1) + 1 = 16t_n + 5$

et $5t_{n+1} - 4t_n = 5\times(4t_n + 1) - 4t_n = 16t_n + 5$

Ainsi, $\forall n \in \mathbb{N}$, $t_{n+2} = 5t_{n+1} - 4t_n$, $t_0 = 0$ et $t_1 = 1$

Donc (u_n) et (t_n) vérifient la même relation de récurrence donc $\forall n,\ u_n = t_n$

b) On pose alors $v_n = u_n + \frac{1}{3}$ pour tout $n \in \mathbb{N}$.

$\forall n \in \mathbb{N}$, $v_{n+1} = u_{n+1} + \frac{1}{3} = 4u_n + 1 + \frac{1}{3} = 4u_n + \frac{4}{3} = 4\left(u_n + \frac{1}{3}\right) = 4v_n$

Donc (v_n) est géométrique de raison 4, de premier terme $v_0 = \frac{1}{3}$.

On a donc $\forall n \in \mathbb{N}$, $v_n = \frac{1}{3} \times 4^n$

c) Ainsi $u_n = v_n - \frac{1}{3}$ et donc $\forall n \in \mathbb{N}$, $u_n = \frac{1}{3} \times 4^n - \frac{1}{3}$

$$u_n = \frac{4^n - 1}{3}$$

Exercice 3

1°) $S_1 = \sum_{k=0}^{n} 5k + 2 + \left(\frac{3}{4}\right)^k = 5\sum_{k=0}^{n} k + \sum_{k=0}^{n} 2 + \sum_{k=0}^{n} \left(\frac{3}{4}\right)^k$

$$= 5 \times \frac{n(n+1)}{2} + (n+1) \times 2 + \frac{1 - \left(\frac{3}{4}\right)^{n+1}}{1 - \frac{3}{4}}$$

$$S_1 = (n+1)\,\frac{5n+4}{2} + 4 \times \left(1 - \left(\frac{3}{4}\right)^{n+1}\right)$$

2°) $\sum_{k=n}^{2n} k = \frac{n + 2n}{2} \times (n+1) = \frac{3(n+1)n}{2}$